Differential Geometry I | Week 12

Last time: We defined principal curvatures, Gauss curvature etc. For a co-oriented surface $S \subset \mathbb{R}^3$.

<u>Theorem</u>: If all the points of a connected surface S of class C^3 are umbilic, then S is contained in a sphere or in a plane.

<u>Proof</u>: Let $p \in S$ and n a co-orientation of S around p. Since S is umbilic, the shape operator (defined with respect to n) takes the form $L_q = \lambda(q) \cdot Id$ (i.e. $\kappa_1(q) = \kappa_2(q)$) for all points q in a neighborhood of p.

Let $\psi : \Omega \to S$ be a local parametrization of S, with $n(u) = n(\psi(u))$ the associated normal $\left(= \frac{b_1 \times b_2}{\|b_1 \times b_2\|} \right)$. Then in view of the above:

$$dn_{\psi(u)}(b_i) := L_{\psi(u)}(b_i) = \overset{\text{"}\lambda(\psi(u))\text{"}}{\lambda(u)} \cdot b_i$$

$$\Rightarrow \frac{\partial}{\partial u_i}(n(u)) = \lambda(u) \cdot \frac{\partial \psi}{\partial u_i} \Rightarrow \frac{\partial^2}{\partial u_i \partial u_j}(n(u)) = \frac{\partial \lambda}{\partial u_j} \cdot \frac{\partial \psi}{\partial u_i} + \lambda \cdot \frac{\partial^2 \psi}{\partial u_i \partial u_j}$$

Repeating the same but with $i \leftrightarrow j$: $\frac{\partial^2}{\partial u_j \partial u_i}(n(u)) = \frac{\partial \lambda}{\partial u_i} \frac{\partial \psi}{\partial u_j} + \lambda \frac{\partial^2 \psi}{\partial u_j \partial u_i}$

Since $n(\psi(u)), \psi$ are C^2 functions: Second derivatives commute, so:

$$\frac{\partial \lambda}{\partial u_i} \cdot \frac{\partial \psi}{\partial u_j} = \frac{\partial \lambda}{\partial u_j} \cdot \frac{\partial \psi}{\partial u_i}, \quad i,j = 1,2.$$

For $i = 1, j = 2$: Since $\frac{\partial \psi}{\partial u_1}, \frac{\partial \psi}{\partial u_2}$ are linearly independent $\Rightarrow \frac{\partial \lambda}{\partial u_1} = \frac{\partial \lambda}{\partial u_2} = 0$

So λ is constant on all of S (since S is connected).

We will consider two cases:

• If $\lambda = 0$: $\frac{\partial}{\partial u_i}(n(u)) = \lambda \cdot b_i = 0$ so $n(u)$ is constant

$\Rightarrow S$ is contained in a plane perpendicular to n

• If $\lambda \neq 0$: $\frac{\partial}{\partial u_i} n(u) = \lambda \cdot \frac{\partial \psi}{\partial u_i} \Rightarrow \frac{\partial}{\partial u_i}(n(u) - \lambda \cdot \psi(u)) = 0$

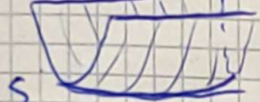
$\Rightarrow n(u) - \lambda \cdot \psi(u) = c$: const

$\Rightarrow n(u) = \lambda \cdot \psi(u) + c$

Since $\|n\| = 1 \Rightarrow \|\lambda \cdot \psi + c\| = 1 \Leftrightarrow \|\psi - (-\frac{c}{\lambda})\| = \frac{1}{|\lambda|}$.

So S lies in the sphere of radius $\frac{1}{|\lambda|}$ centered at $-\frac{c}{\lambda}$ ☒

The second fundamental form (or the shape operator):
Determines the extrinsic geometry of a surface
(i.e. the way it bends in space).



However, Gauss discovered in 1827 that the product $\kappa_1 \cdot \kappa_2$ (the Gauss curvature) is determined only by the intrinsic geometry (and can thus be determined by measuring lengths, angles etc intrinsically in the surface).

<u>Theorem</u> (Theorema Egregium = "Remarkable theorem):
Let $\psi : \Omega \to S$ be a C^3 local parametrization of the surface.
Then K (the Gauss curvature) can be expressed in terms of g_{ij} (the components of the metric tensor), involving terms of up to 2 derivatio of g.

As a result:

Corollary:

If $f: S_1 \to S_2$ is an isometry of class C^3 between surfaces,

then $K_1 = K_2 \circ f$

Remarks:

- In general, thinking of surfaces that are made of ~~~~ ~~~~ rigid material (e.g. metal sheets, paper), i.e. material that cannot be internally deformed, an isometry of a surface corresponds to a "bending" of that surface (i.e. curving a piece of paper into a cylinder or a cone).

  The above theorem gives a restriction on what surfaces made of such material can be deformed into: A plane can be isometrically mapped to a cylinder, but not to a sphere (hence we cannot have planar maps of the earth that represent distances faithfully).

  If S is (locally) isometric to plane: $K = 0 \Rightarrow$ One principal curvature (at least) is 0 at every point.

  This gives structural stability: Curving a piece of paper in one direction (i.e. ~~~~ introducing a positive principal direction) forces the perpendicular direction to have $\kappa = 0$; the surface cannot be further bent in the perpendicular direction without tearing and crumbling.

- The converse is not in general true: A partial converse (Minding's theorem) says that if $K_1 = K_2 =$ const, then there exists a local isometry between them.

  (This can be easily proved in "normal coordinates")
- Also: Not true if f is not C^2 (counter-intuitive examples by Nash)

In order to prove the Egregium theorem, we have to do a little digression.

<u>Christoffel symbols</u>:

Let $\psi : \Omega \to S \subset \mathbb{R}^3$ be a C^2 local parametrization and let

$$b_1 = \frac{\partial \psi}{\partial u_1}, \quad b_2 = \frac{\partial \psi}{\partial u_2}, \quad n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|}$$

At every $p = \psi(u)$: $\{b_1, b_2, n\}$ span $\mathbb{R}^3$, $\{b_1, b_2\}$ span T_pS.

We can decompose $\frac{\partial^2 \psi}{\partial u_i \partial u_j}$ in this basis: (note that the projection on n, namely $\langle \frac{\partial^2 \psi}{\partial u_i \partial u_j}, n \rangle$, is the second fundamental form):

$$\frac{\partial^2 \psi}{\partial u_i \partial u_j} = \Gamma_{ij}^1 b_i + \Gamma_{ij}^2 b_j + h_{ij} n$$

The components Γ_{ij}^k: The <u>Christoffel symbols</u> of the parametrization.

Let us set $\Gamma_{ijk} = \langle \frac{\partial^2 \psi}{\partial u_i \partial u_j}, \frac{\partial \psi}{\partial u_k} \rangle = \langle \frac{\partial^2 \psi}{\partial u_i \partial u_j}, b_k \rangle$

Then $\Gamma_{ijk} = \langle \Gamma_{ij}^1 b_1 + \Gamma_{ij}^2 b_2 + h_{ij} n, b_k \rangle \overset{n \perp b_1, b_2}{=} \Gamma_{ij}^1 \langle b_1, b_k \rangle + \Gamma_{ij}^2 \langle b_2, b_k \rangle$

$$\Rightarrow \boxed{\Gamma_{ijk} = \sum_{l=1}^{2} \Gamma_{ij}^l g_{lk}}$$

If we denote with g^{ab} the components of the <u>inverse</u> matrix of g_{ij}:

$$\Gamma_{ij}^l = \sum_{k=1}^{2} \Gamma_{ijk} g^{kl}$$

We will call both Γ_{ij}^k and Γ_{ijk} the Christoffel symbols (the ones can be obtained from the other).

So: • h_{ij} : Projection of $\frac{\partial^2 \varphi}{\partial u_i \partial u_j}$ on n

• Γ_{ijk}: Projection of $\frac{\partial^2 \varphi}{\partial u_i \partial u_j}$ on b_k (so on $T_{\varphi(u)} S$).

• Since $\frac{\partial^2 \varphi}{\partial u_i \partial u_j} = \frac{\partial^2 \varphi}{\partial u_j \partial u_i} \Longrightarrow \boxed{\Gamma_{ij}^k = \Gamma_{ji}^k \text{ and } \Gamma_{ijk} = \Gamma_{jik} \quad \text{(symmetry)}}$

The Christoffel symbols can be determined from an expression of the metric tensor g and it's first derivatives:

<u>Lemma</u> (Levi-Civita) : $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$

<u>Remark</u>: The Christoffel symbols are not geometric; they change under changes of parametrization.

<u>Proof</u>: By definition, $g_{ab} = \langle b_a, b_b \rangle = \langle \frac{\partial \varphi}{\partial u_a}, \frac{\partial \varphi}{\partial u_b} \rangle$

Differentiating: $\partial_\gamma g_{\alpha\beta} = \frac{\partial}{\partial u_\gamma} \langle \frac{\partial \varphi}{\partial u_\alpha}, \frac{\partial \varphi}{\partial u_\beta} \rangle$

$$= \langle \frac{\partial^2 \varphi}{\partial u_\gamma \partial u_\alpha}, \frac{\partial \varphi}{\partial u_\beta} \rangle + \langle \frac{\partial \varphi}{\partial u_\alpha}, \frac{\partial^2 \varphi}{\partial u_\gamma \partial u_\beta} \rangle$$

$$= \Gamma_{\gamma\alpha\beta} + \Gamma_{\gamma\beta\alpha}$$

So $\frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = \frac{1}{2}(\Gamma_{ijk} + \cancel{\Gamma_{ikj}} + \Gamma_{jik} + \cancel{\Gamma_{jki}} - \cancel{\Gamma_{kij}} - \cancel{\Gamma_{kji}})$

$= \Gamma_{ijk}$ (due to symmetry) $\square$

Proof of the Egregium theorem:

By definition, $K = \det L = \frac{\det H}{\det G} = \frac{h_{11} h_{22} - h_{12}^2}{g_{11}\cdot g_{22} - g_{12}^2}$

So we need to show that $h_{11}\cdot h_{22} - h_{12}^2$ can be expressed only in terms of derivatives of g of order up to 2.

Note that we have $h_{ij} = \langle \frac{\partial^2\psi}{\partial u_i \partial u_j}, n \rangle$

So let's compute: We will start with the following identity

$$\left\langle \frac{\partial^2\psi}{\partial u_i\partial u_j}, \frac{\partial^2\psi}{\partial u_\alpha\partial u_\beta} \right\rangle = \langle \Gamma_{ij}^1 b_1 + \Gamma_{ij}^2 b_2 + h_{ij} n, \Gamma_{\alpha\beta}^1 b_1 + \Gamma_{\alpha\beta}^2 b_2 + h_{\alpha\beta} n \rangle$$

$$\overset{b_i \perp n}{\underset{\langle b_i, b_j\rangle = g_{ij}}{=}} \Gamma_{ij}^1 \Gamma_{ab}^1 g_{11} + \Gamma_{ij}^1 \Gamma_{ab}^2 g_{12} + \Gamma_{ij}^2 \Gamma_{ab}^1 g_{12} + \Gamma_{ij}^2 \Gamma_{ab}^2 g_{22} + h_{ij} h_{ab}$$

$$= \Gamma_{ij}^1 \Gamma_{ab1} + \Gamma_{ij}^2 \Gamma_{ab2} + h_{ij} h_{ab}$$

$$\Rightarrow \langle \partial_{ij}^2 \psi, \partial_{ab}^2 \psi \rangle = \sum_{l=1}^{2} \Gamma_{ij}^l \Gamma_{abl} + h_{ij} h_{ab} \quad (1)$$

Also: $\partial_1 \Gamma_{221} - \partial_2 \Gamma_{121} = \partial_1 \langle \partial_{22}^2 \psi, \partial_1 \psi \rangle - \partial_2 \langle \partial_{12}^2 \psi, \partial_1 \psi \rangle$

$$= \cancel{\langle \partial_{122}^3 \psi, \partial_1 \psi \rangle} + \langle \partial_{22}^2 \psi, \partial_{11}^2 \psi \rangle - \cancel{\langle \partial_{212}^3 \psi, \partial_1 \psi \rangle} - \langle \partial_{12}^2 \psi, \partial_{12}^2 \psi \rangle$$

So from (1): $\partial_1 \Gamma_{221} - \partial_2 \Gamma_{121} = \sum_{l=1}^{2} \left(\Gamma_{11}^l \Gamma_{22l} - \Gamma_{12}^l \Gamma_{12l} \right) + h_{11} h_{22} - h_{12}^2$

$$\Longrightarrow h_{11}\cdot h_{22} - h_{12}^2 = \partial_1 \Gamma_{221} - \partial_2 \Gamma_{121} - \sum_{l=1}^{2} \left(\Gamma_{11}^l \Gamma_{22l} - \Gamma_{12}^l \Gamma_{12l} \right)$$

Since $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta} \right)$: QED $\boxed{}$

Note: In the above expression

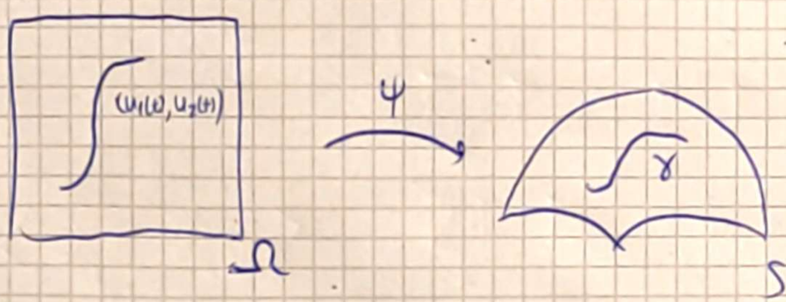
$$K = \frac{1}{g_{11}g_{22} - g_{12}^2}\left(\partial_1 \Gamma_{221} - \partial_2 \Gamma_{121} - \sum_{l=1}^{2}\left(\Gamma_{11}^{l}\,\Gamma_{22l} - \Gamma_{12}^{l}\,\Gamma_{12l}\right)\right)$$

the left hand side: geometric (independent of parametrization)

The right hand side: Consists of individual pieces which are not separately invariant of parametrization (the total expression: "Scalar curvature" From Riemannian geometry).

The Christoffel symbols: Actually appear in the calculation of the intrinsic acceleration (i.e. geodesic curvature) of curves in a surface:

Let $\psi : \Omega \to S$ be a C^2 local parametrization of a surface



For any curve $\gamma : I \to S$. Let $(u_1(t), u_2(t)) = \psi^{-1} \circ \gamma(t)$ be the representation of γ in the (u_1, u_2) coordinates

(so that $\gamma(t) = \psi(u_1(t), u_2(t)) = \psi(u(t))$)

Then by the composition rule for derivatives:

$$\dot{\gamma}(t) = \frac{d}{dt}(\psi(u(t))) = \frac{\partial \psi}{\partial u_1}\cdot \dot{u}_1 + \frac{\partial \psi}{\partial u_2}\cdot \dot{u}_2$$

$$\Rightarrow \ddot{\gamma}(t) = \frac{\partial \psi}{\partial u_1}\cdot \ddot{u}_1 + \frac{\partial \psi}{\partial u_2}\cdot \ddot{u}_2 + \frac{\partial^2 \psi}{\partial u_1^2}\dot{u}_1\dot{u}_1 + 2\frac{\partial^2 \psi}{\partial u_1 \partial u_2}\dot{u}_1\dot{u}_2 + \frac{\partial^2 \psi}{\partial u_2^2}\dot{u}_2\dot{u}_2$$

From the expression for the Hessian of ψ:

$$\frac{\partial^2 \psi}{\partial u_i \partial u_j} = \Gamma_{ij}^1 b_1 + \Gamma_{ij}^2 b_2 + h_{ij} n$$

We get

$$\left.\begin{aligned}\ddot{\gamma}(t) = &\left(\ddot{u}_1 + \Gamma_{11}^1 \dot{u}_1^2 + 2\Gamma_{12}^1 \dot{u}_1 \dot{u}_2 + \Gamma_{22}^1 \dot{u}_2^2\right) b_1 \\ &+ \left(\ddot{u}_2 + \Gamma_{11}^2 \dot{u}_1^2 + 2\Gamma_{12}^2 \dot{u}_1 \dot{u}_2 + \Gamma_{22}^2 \dot{u}_2^2\right) b_2\end{aligned}\right\} \text{tangential part to } S$$

$$+ \underbrace{\left(h_{11} \dot{u}_1^2 + 2h_{12} \dot{u}_1 \dot{u}_2 + h_{22} \dot{u}_2^2\right)}_{h(\dot{\gamma},\dot{\gamma})} \hat{n}$$

So: $\langle n, \ddot{\gamma} \rangle = h(\dot{\gamma},\dot{\gamma})$ (recall Meusnier's theorem)

And for γ to be a geodesic: $\ddot{\gamma} \,//\, n$

$$\Leftrightarrow \begin{cases} \ddot{u}_1 + \sum_{i,j=1}^{2} \Gamma_{ij}^1 \dot{u}_i \dot{u}_j = 0 \\ \ddot{u}_2 + \sum_{i,j=1}^{2} \Gamma_{ij}^2 \dot{u}_i \dot{u}_j = 0 \end{cases}$$

"Internal" description of a geodesic: 2^{nd} order system of ODEs

<u>The Gauss-Codazzi equations</u>

These equations relate the 2^{nd} fundamental form with the intrinsic geometry.

<u>Gauss eqn</u>: $h_{11} h_{22} - h_{12}^2 = \frac{1}{\det(g)}\left(\partial_1 \Gamma_{221} - \partial_2 \Gamma_{121} - \sum_{l=1}^{2}\left(\Gamma_{11}^l \Gamma_{22l} - \Gamma_{12}^l \Gamma_{12l}\right)\right)$

<u>Codazzi eqn</u>: $\partial_1 h_{21} - \partial_2 h_{11} - \sum_{l=1}^{2}\left(\Gamma_{11}^l h_{l2} - \Gamma_{21}^l h_{l1}\right) = 0$

$$\partial_1 h_{22} - \partial_2 h_{12} - \sum_{l=1}^{2}\left(\Gamma_{12}^l h_{l2} - \Gamma_{22}^l h_{l1}\right) = 0$$

(3 eqns for 3 unknowns: h_{11}, $h_{12} = h_{21}$, h_{22}. The system is <u>elliptic</u> when $K>0$, hyperbolic when $K<0$

Proof of the Codazzi eqn:

Start from $\dfrac{\partial^2 \psi}{\partial u_i \partial u_j} = \sum_{l=1}^{2} \Gamma_{ij}^{l} \dfrac{\partial \psi}{\partial u_l} + h_{ij}\, n$

Differentiate: $$\frac{\partial^3 \psi}{\partial u_a \partial u_i \partial u_j} = \sum_{l=1}^{2} (\partial_a \Gamma_{ij}^{l}) \underbrace{\frac{\partial \psi}{\partial u_l}}_{\partial_l \psi} + \sum_{l=1}^{2} \Gamma_{ij}^{l} (\partial^2_{al} \psi) + (\partial_a h_{ij})\, n + h_{ij}\, \partial_a n$$

Note: $\partial_a \psi = b_a \perp n$ and $\partial_a n \perp n$ (since $\langle n, n \rangle = 1$)

So taking the inner product with n:

$$\langle \partial^3_{ail} \psi, n \rangle = \sum_{l=1}^{2} \Gamma_{ij}^{l} \underbrace{\langle \partial^2_{al} \psi, n \rangle}_{h_{al}} + \partial_a h_{ij}$$

Since $\partial^3_{ail} \psi = \partial^3_{ial} \psi$:

$$0 = \langle \partial^3_{ail} \psi, n \rangle - \langle \partial^3_{ial} \psi, n \rangle = \sum_{l=1}^{2} \left(\Gamma_{ij}^{l} h_{al} - \Gamma_{aj}^{l} h_{il} \right) + \partial_a h_{il} - \partial_i h_{al}.$$

$\square$

Rigidity of surfaces

Definition: Let S, S' be two surfaces in $\mathbb{R}^3$. We will say that S and S' are related by a rigid motion if

$S' = F(S)$, where $F: \mathbb{R}^3 \to \mathbb{R}^3$ is of the form

$F(x) = Mx + b$; $M \in SO(3)$ (i.e. F is an isometry of $\mathbb{R}^3$ preserving the orientation).

Let $\psi : \Omega \to S$ and $\psi' : \Omega \to S'$

be two parametrizations (note: from the same domain Ω) such that $\psi' = F \circ \psi$, where $F(x) = Mx + b$, $M \in SO(3)$.

Then $g_{ij} = g'_{ij}$, $h_{ij} = h'_{ij}$.

<u>Proof</u>: Since $\psi' = M \cdot \psi + b \overset{M, b \text{ - constant}}{\Longrightarrow} \frac{\partial \psi'}{\partial u_i} = M \cdot \frac{\partial \psi}{\partial u_i}$

(or $b'_i = M \cdot b_i$). Since $M \in SO(3)$: $Mb_1 \times Mb_2 = M(b_1 \times b_2)$,

so $n' = Mn$

Moreover: $\frac{\partial^2 \psi'}{\partial u_i \partial u_j} = M \cdot \frac{\partial^2 \psi}{\partial u_i \partial u_j}$

Hence: $g'_{ij} = \langle b'_i, b'_j \rangle = \langle Mb_i, Mb_j \rangle = \langle b_i, b_j \rangle$ (since $M \in SO(3)$)
$= g_{ij}$

$h'_{ij} = \langle \frac{\partial^2 \psi'}{\partial u_i \partial u_j}, n' \rangle = \langle M \frac{\partial^2 \psi}{\partial u_i \partial u_j}, Mn \rangle = \langle \frac{\partial^2 \psi}{\partial u_i \partial u_j}, n \rangle = h_{ij}$ $\square$

The converse is also true: (if Ω is connected)

<u>Proposition</u>: Let $\psi : \Omega \to S$ and $\psi' : \Omega \to S'$ be C^2 local parametrizations such that $g'_{ij}(u) = g_{ij}(u)$ and $h'_{ij}(u) = h_{ij}(u)$ for all $u \in \Omega$, $i, j \in \{1, 2\}$. Then there exists an (affine) isometry $F : \mathbb{R}^3 \to \mathbb{R}^3$ preserving the orientation such that $\psi' = F \circ \psi$

<u>Note</u>: In the above, we have assumed that we have fixed the coorientation $n = \frac{b_i \times b_j}{\|b_i \times b_j\|}$ (and similarly for n')

Proof:

For any $u \in \Omega$, let $M(u): \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map mapping the basis $\{b_1(u), b_2(u), n(u)\}$ to $\{b_1'(u), b_2'(u), n'(u)\}$

(i.e. $b_1' = M \cdot b_1$, $b_2' = M \cdot b_2$, $n' = Mn$. note M depends on $u \in \Omega$!)

Since $g_{ij}' = g_{ij} \Rightarrow \langle b_i'(u), b_j'(u) \rangle = \langle b_i(u), b_j(u) \rangle$

$$\Rightarrow \langle M b_i(u), M b_j(u) \rangle = \langle b_i(u), b_j(u) \rangle$$

Also: $\langle n', b_i' \rangle = 0 = \langle n, b_i \rangle$

So $\langle Mn, Mb_i \rangle = \langle n, b_i \rangle$, $i=1,2$

and $\langle n', n' \rangle = 1 = \langle n, n \rangle$, so

$$\langle Mn, Mn \rangle = \langle n, n \rangle$$

From the above: $\forall\ v \in \mathbb{R}^3$, splitting $v = v^{(1)} b_1 + v^{(2)} b_2 + v^{(3)} n$,

we see that $\langle Mv, Mv \rangle = \langle v, v \rangle \Rightarrow M(u) \in O(3)\ \forall u \in \Omega$

and, since $n' = \dfrac{b_1' \times b_2'}{\|b_1' \times b_2'\|}$ and $n = \dfrac{b_1 \times b_2}{\|b_1 \times b_2\|}$: The bases $\{b_1, b_2, n\}$

and $\{b_1', b_2', n'\}$ are both positively oriented $\Rightarrow \det M > 0$

So $\forall u \in \Omega$: $M(u) \in SO(3)$.

We will show that $M(u)$ is a constant matrix:

Note $\dfrac{\partial^2 \psi'}{\partial u_i \partial u_j} = \dfrac{\partial}{\partial u_i} b_j' = \dfrac{\partial}{\partial u_i}(M(u)\, b_j(u)) = \dfrac{\partial M}{\partial u_i} \cdot b_j + M \cdot \dfrac{\partial}{\partial u_i} b_j$

$$= \frac{\partial M}{\partial u_i} b_j + M \cdot \frac{\partial^2 \psi}{\partial u_i \partial u_j} \quad ①$$

However:

Since $g'_{ij} = g_{ij}$ (so also $\Gamma'_{ijk} = \Gamma_{ijk}$) and $h_{ij} = h'_{ij}$:

$$\frac{\partial^2 \psi}{\partial u_i \partial u_j} = \Gamma^1_{ij} b_1 + \Gamma^2_{ij} b_2 + h_{ij} n$$

$$\frac{\partial^2 \psi'}{\partial u_i \partial u_j} = (\Gamma^1_{ij})' b'_1 + (\Gamma^2_{ij})' b'_2 + h'_{ij} n'$$

$$= \Gamma^1_{ij} b'_1 + \Gamma^2_{ij} b'_2 + h_{ij} n'$$

$$= \Gamma^1_{ij} \cdot M b_1 + \Gamma^2_{ij} \cdot M b_2 + h_{ij} \cdot M n'$$

$$= M(u) \cdot \frac{\partial^2 \psi}{\partial u_i \partial u_j}$$

So returning to ①: $\frac{\partial M}{\partial u_i} = 0$, $i = 1,2$ $\Rightarrow$ M is constant.

So: $\frac{\partial \psi'}{\partial u_i} = M \cdot \frac{\partial \psi}{\partial u_i}$ $\Rightarrow$ $\psi'(u) = M \cdot \psi(u) + b$ ▨